

UNCLASSIFIED

AD - 2 2 4 3 8 5

DEFENSE DOCUMENTATION CENTER

FOR

SCIENTIFIC AND TECHNICAL INFORMATION

CAMERON STATION, ALEXANDRIA, VIRGINIA



UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

AD No. 224385

DDC FILE COPY

UNCLASSIFIED

DDC

6 BEST EXPIORATION FOR MAXIMUM IS FIBONACCIAN,

15 by Selmer M. Johnson.

P-856

May 4, 1956

734 200

 **RAND** Corporation

1700 MAIN ST. • SANTA MONICA, CALIF. 90401

#### SUMMARY

A unimodal function of one variable is defined on an interval. No regularity conditions involving continuity, derivatives, etc., are assumed. We wish to minimize the number of calculations of values of the function in order to assure the location of its maximum to a prescribed degree of accuracy. The solution of this problem and its discrete analogue involves the well-known Fibonacci sequence.

## BEST EXPLORATION FOR MAXIMUM IS FIBONACCIAN

Selmer M. Johnson

When the author obtained the result presented here, he thought it was new. Subsequently, it was found that the result had previously been published by J. Kiefer [1]. However, the simplicity of the present proof may justify a short note on the problem.

In some practical problems the only way to locate a maximum of a function is to observe the values of the function at various points and compare them. For example, the function may not be continuous, or perhaps the derivatives may be too hard to find analytically, etc. With this in mind, and as a first step toward the solution of the general problem, we consider the case of a unimodal function on an interval.

**Definition:** A function  $f(x)$  is unimodal if there is an  $x_0$  such that  $f$  is either strictly increasing for  $x \leq x_0$  and strictly decreasing for  $x > x_0$ , or else strictly increasing for  $x < x_0$  and strictly increasing for  $x \geq x_0$ .

For example, concave functions are unimodal.

In [1], Kiefer considers the problem of determining an interval containing the point at which a unimodal function on the unit interval possesses a maximum. No regularity conditions

concerning continuity or derivatives, etc., are assumed. He gives, for every  $\epsilon > 0$  and every number  $n$  of values of the argument at which the function may be observed, a procedure which is  $\epsilon$ -minimax among the class of all sequential non-randomized procedures which terminate by giving an interval containing the required point, where the payoff is the length of the final interval.

We present a short proof of the result after it has been restated as follows.

Theorem 1. Let  $y = f(x)$  be any unimodal function defined on an interval  $0 \leq x \leq L_n$ . Let  $F_n = \sup$  of all  $L_n$  with the property that we can always locate the maximum of  $f(x)$  on a unit-length subinterval by calculating  $n$  values of the function. Then  $F_n$  is the  $n$ -th Fibonacci number; that is,

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2,$$

with  $F_0 = F_1 = 1$ .

The proof will be by induction.

We observe first that  $F_0 = F_1 = 1$  follows from the definition of  $F_n$ , since just one value of the function gives no information concerning the location of the maximum.

Fix  $n \geq 2$  and calculate  $y_1 = f(x_1)$ ,  $y_2 = f(x_2)$ , where  $0 < x_1 < x_2 < L_n$ . If  $y_1 > y_2$ , the maximum occurs on  $(0, x_2)$  since  $f(x)$  is strictly unimodal. If  $y_2 > y_1$ , the maximum is

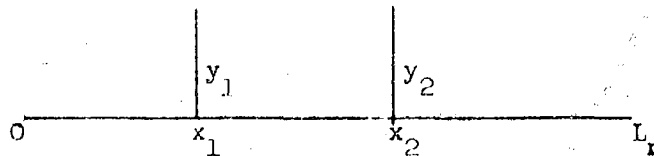
on  $(x_1, L_n)$ . If  $y_1 = y_2$ , choose either of the above intervals, even though we know the maximum occurs on  $(x_1, x_2)$ . Thus at each stage after the first computation we are left with a subinterval and the value of  $f(x)$  at some interior point  $x$ . Since values at the ends of an interval do not always give usable information relative to our problem, we restrict our attention to the interior points.

For  $n = 2$ ,  $L_n = 2 - \epsilon$ , take  $x_1 = 1 - \epsilon$ ,  $x_2 = 1$ , for arbitrarily small  $\epsilon > 0$ . By the argument of the preceding paragraph, we have  $F_2 = 2 = F_1 + F_0$ .

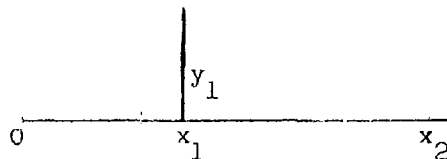
Fix  $n > 2$ , and assume that  $F_k = F_{k-1} + F_{k-2}$  for  $k = 2, \dots, n-1$ . We shall show that

$$F_n = F_{n-1} + F_{n-2}.$$

For if we calculate at  $x_1$  and  $x_2$  on  $(0, L)$  we have the picture



If  $y_1 > y_2$ , we have the new picture



But then  $x_2 < F_{n-1}$  since we have only  $n - 2$  more choices with  $x_1$  a first choice for the case  $k = n - 1$ . Moreover,



$x_1 < F_{n-2}$ , since the maximum could occur on  $(0, x_1)$  with two choices of  $x$  already used.

Similarly if  $y_2 > y_1$ , we have  $L_n - x_1 < F_{n-1}$ . Thus  $L_n < F_{n-1} + x_1 < F_{n-1} + F_{n-2}$ , so that  $F_n < F_{n-1} + F_{n-2}$ .

Now choose  $L_n + (1 - \frac{\epsilon}{2})(F_{n-1} + F_{n-2})$ ,  $x_1 + (1 - \frac{\epsilon}{2})F_{n-2}$ , and  $x_2 = (1 - \frac{\epsilon}{2})F_{n-1}$ . Since  $\epsilon$  is arbitrarily small this shows that  $F_n = F_{n-1} + F_{n-2}$ , so Theorem 1 is proved. Moreover, the procedure is optimal for a given  $\epsilon$ , since after comparing  $f(x_1)$  and  $f(x_2)$  we are left with an interval of length  $L_{n-1} = (1 - \frac{\epsilon}{2})F_{n-1}$ , and with a value at optimal first position for this smaller interval. Continuing in this way we have  $L_k = (1 - \frac{\epsilon}{2})F_k$  for  $k \leq n$ ; in particular,  $L_2 = (1 - \frac{\epsilon}{2})F_2 = 2 - \epsilon$ , and the final interval is of unit length.

The  $F_n$  form the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots,$$

with  $F_{20} > 10000$ . Thus a maximum can always be located within  $10^{-4}$  of the original interval length with 20 calculations. Using the more conventional technique of computing  $f(x)$  and  $f(x + \epsilon)$ , where  $x$  is at the midpoint of the interval, may take as many as 28 computations.

As Kiefer remarks, since  $F_{n+1}/F_n$  rapidly approaches its limit  $\tau = \frac{1}{2}(\sqrt{5} + 1) = 1.618$ , one could choose the first two values of  $x$  at  $L/\tau = .618 L$  from either end of the interval. This would be an excellent approximation at each stage except for the

last few choices, especially if  $n$  is not fixed in advance but is determined after several values have been observed (e.g., more observations might be taken if the function appears to be sharply peaked near its maximum). This constant ratio to determine the next  $x$  would be useful in a computing-machine code.

A line segment is said to be divided according to the "golden section" if one part is  $\tau$  times the other. The approximation suggested above gives at each stage an interval which is divided by the golden section at its  $x$  value but reduced in scale from the previous interval. This follows from the relation

$$\frac{L / \tau}{L(1 - \frac{1}{\tau})} = \frac{1}{\tau - 1} = \tau$$

#### The Discrete Case

Theorem 2. Let  $y = f(x)$  be any unimodal function defined on a discrete set of  $H_n$  points. Let  $K_n = \max H_n$  such that the function's maximum can always be identified in  $n$  observations. Then

$$K_n = -1 + F_{n+1}, \quad n \geq 1$$

Proof. Number the points 1, 2, 3, ...,  $H_n$ . Observe that  $K_1 = 1$ ,  $K_2 = 2$ ,  $K_3 = 4$ . Fix  $n > 3$  and assume  $K_k = -1 + F_{k+1}$  for  $k < n$ . Calculate  $f(x)$  at  $x_1$  and  $x_2$ . By arguments analogous

to those of Theorem 1, we must have

$$x_1 \leq K_{n-2} + 1, \quad H_n - x_1 \leq K_{n-1},$$

so that

$$H_n \leq K_{n-2} + 1 + K_{n-1} = (F_{n-1} - 1) + 1 + F_n - 1 = F_{n+1} - 1.$$

This maximum is attained when  $x_1 = F_{n-1}$  and  $x_2 = F_n$ . Thus Theorem 2 is proved; moreover, the optimal construction is given.

#### REFERENCE

1. Kiefer, J., "Sequential Minimax Search for a Maximum,"  
Proceedings of the American Mathematical Society,  
Vol. 4, No. 3, June, 1953, pp. 502-506.

UNCLASSIFIED

UNCLASSIF